Zeta functions of groups and rings

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Abstract. We report on progress and problems concerning the analytical behaviour of the zeta functions of groups and rings. We also describe how these generating functions are special cases of adelic cone integrals for which our results hold.

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1. Introduction

Whenever a counting problem pertaining to some mathematical object Λ produces a sequence of non-negative integers $a_\Lambda(n)$ $(n = 1, 2, \dots)$ we can hope to gain information by incorporating our sequence into a generating function. There are various ways of doing this, for example as coefficients of power series, sums representing automorphic functions and Dirichlet series. Sometimes there is a natural choice of a generating function dictated by the recursive properties of the sequence $a_\Lambda(n)$. We report here on counting problems where the choice of a Dirichlet series seems to be appropriate.

We consider first two counting problems relating to a finitely generated group G. Write $[G : H]$ for the index of a subgroup $H \le G$ and let

$$a_G^{<}(n) := |\{ H \le G \mid [G:H] = n \}|, \quad a_G^{\triangleleft}(n) := |\{ H \trianglelefteq G \mid [G:H] = n \}| \quad (1)$$

be the number of subgroups or normal subgroups of index precisely n in G. The numbers $a_G^{<}(n)$ all being finite we call

$$\zeta_G^{<}(s) := \sum_{n=1}^{\infty} a_G^{<}(n)\, n^{-s} = \sum_{H \le_f G} [G:H]^{-s} \quad (2)$$

the *subgroup zeta function* of G. The symbol $H \le_f G$ indicates that the summation is over all subgroups H of finite index in G. Similarly, we define

$$\zeta_G^{\triangleleft}(s) := \sum_{n=1}^{\infty} a_G^{\triangleleft}(n)\, n^{-s} = \sum_{H \trianglelefteq_f G} [G:H]^{-s} \quad (3)$$

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to be the *normal subgroup zeta function* of G. When we intend to address both types of zeta functions simultaneously we write $\zeta_G^*(s)$ for $\zeta_G^{\leq}(s)$ or $\zeta_G^{\lhd}(s)$.

For the second type of counting problem we consider a ring R, which is for our purposes an abelian group R carrying a biadditive product. Let us write $S \leq R$ if S is a subring of R and $\mathfrak{a} \trianglelefteq R$ if $\mathfrak{a}$ is a left ideal of R. Let

$$a_R^{\leq}(n) := |\{ S \leq R \mid [R : S] = n \}|, \quad a_R^{\lhd}(n) := |\{ \mathfrak{a} \trianglelefteq R \mid [R : \mathfrak{a}] = n \}| \tag{4}$$

be the numbers of these subobjects of R which have index $n \in \mathbb{N}$ in the additive group of R. The numbers counting subrings are finite if the additive group of R is finitely generated. The numbers counting ideals are all finite under the weaker hypothesis that the ring R is finitely generated. Given these circumstances define the *subring zeta function* or the *ideal zeta function* to be respectively

$$\zeta_R^{\leq}(s) := \sum_{n=1}^{\infty} a_R^{\leq}(n)\, n^{-s}, \quad \zeta_R^{\lhd}(s) := \sum_{n=1}^{\infty} a_R^{\lhd}(n)\, n^{-s}. \tag{5}$$

Again we write $\zeta_R^*(s)$ for $\zeta_R^{\leq}(s)$ or $\zeta_R^{\lhd}(s)$.

While the study of the zeta functions of a finitely generated group was only begun in [27], the ideal zeta function of a ring has the Riemann zeta function ($R = \mathbb{Z}$) or more generally the Dedekind zeta function of the ring of integers in a number field as special cases (see [32]).

We wish to consider the Dirichlet series (2), (3), (5) not only as formal sums but as series converging in a non-empty subset of the complex numbers. By general theory this subset may be taken to be a right half-plane. In fact, this convergence condition will be satisfied if and only if the coefficients in the series (2), (3), (5) grow at most polynomially in n, more precisely if and only if there are $t,\ c^* \in \mathbb{R}$ such that $a_G^*(n) \leq c^* n^t$ respectively $a_R^*(n) \leq c^* n^t$ holds for all $n \in \mathbb{N}$. In this case we will say that G has *polynomial subgroup* or *normal subgroup growth*, the ring R will be said to have *polynomial subring* or *ideal growth*. For finitely generated groups G there is the following beautiful characterisation of this property by A. Lubotzky, A. Mann and D. Segal (see [36]).

Theorem 1.1. *Let G be a finitely generated residually finite group. Then G has polynomial subgroup growth if and only if G has a subgroup of finite index which is soluble and of finite rank.*

A group G is called residually finite if for every non-trivial $g \in G$ there is a subgroup H of finite index in G with $g \notin H$. Of course, this assumption is natural for Theorem 1.1 to hold. A group G is said to be of finite rank $r \in \mathbb{N}$ if every finitely generated subgroup of G can be generated by at most r elements.

In the following we shall assume that

- either $\Lambda = G$ is an infinite finitely generated torsion-free nilpotent group,
- or $\Lambda = R$ is a ring with additive group isomorphic to $\mathbb{Z}^d$ for some $d \in \mathbb{N}$.

We shall denote the set of isomorphism classes of such objects by $\mathcal{T}$.

Finitely generated torsion-free nilpotent groups satisfy the growth condition in Theorem 1.1. Their classification up to isomorphism is intimately connected with the reduction theory for arithmetic groups (see [24]). In addition there are connections between special classes of nilpotent groups and certain diophantine problems including the question of equivalence classes of integral quadratic forms ([25]). See [26] for a panorama of finitely generated torsion-free nilpotent groups. The class of rings in $\mathcal{T}$ contains all rings of integers in number fields and also (for example) the integer versions of the simple Lie algebras over $\mathbb{C}$.

For $\Lambda \in \mathcal{T}$ the zeta functions share a number of features in common with the Dedekind zeta function of a number field. Before we report the story let us mention some examples. Considering $\mathbb{Z}^d$ ($d \in \mathbb{N}$) as a direct product of infinite cyclic groups we find $\zeta^*_{\mathbb{Z}^d}(s) = \zeta(s)\zeta(s-1)\dots\zeta(s-d+1)$ where

$$\zeta(s) = \zeta^{<}_{\mathbb{Z}}(s) = \zeta^{\triangleleft}_{\mathbb{Z}}(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p \frac{1}{1-p^{-s}}$$

is the Riemann zeta function. A more elaborate example concerns the discrete Heisenberg group H_3, that is the group of strictly upper triangular 3×3-matrices with integer entries. The group H_3 is a torsion-free, nilpotent group of class 2 generated by two elements. The following formulas are proved in [46], see also [27].

$$\zeta^{<}_{H_3}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}, \quad \zeta^{\triangleleft}_{H_3}(s) = \zeta(s)\zeta(s-1)\zeta(3s-2). \tag{6}$$

For an interesting example of the zeta function of a ring we can consider $sl_2(\mathbb{Z})$, the additive group of integer 2×2-matrices of trace 0 with the usual Lie bracket. The following formula was finally proved in [21] after contributions in [29], [5], [6]

$$\zeta^{<}_{sl_2(\mathbb{Z})}(s) = P(2^{-s})\frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-1)}{\zeta(3s-1)} \tag{7}$$

where $P(x)$ is the rational function $P = (1+6x^2-8x^3)/(1-x^3)$.

All these examples of zeta functions of members of $\mathcal{T}$ have three distinctive properties (evident from the formulas given):

- they converge in some right halfplane of $\mathbb{C}$,
- they decompose similarly to the Riemann or Dedekind zeta function as an Euler product of some rational expression in p^{-s} taken over all primes p,
- they have a meromorphic continuation to $\mathbb{C}$.

We believe that these three properties already justify the name zeta function for the corresponding generating function. A fourth property of the Dedekind zeta function, the global functional equation (see [32]) is hardly conceivable looking at formulas (6), (7).

Let us define now what will be the Euler factors of the zeta function of a general $\Lambda \in \mathcal{T}$. For a prime p we set:

$$\zeta^*_{\Lambda,p}(s) := \sum_{k=0}^{\infty} a^*_\Lambda(p^k)\, p^{-ks}. \tag{8}$$

This expression can be considered as a function in the variable $s \in \mathbb{C}$ or equally as a power series in p^{-s}.

In [27] the following theorem is established.

Theorem 1.2. *For $\Lambda \in \mathcal{T}$ the following hold.*

(i) *The Dirichlet series $\zeta^*_\Lambda(s)$ converges in some right half-plane of $\mathbb{C}$.*

(ii) *The Dirichlet series $\zeta^*_\Lambda(s)$ decomposes as an Euler product*

$$\zeta^*_\Lambda(s) = \prod_p \zeta^*_{\Lambda,p}(s), \tag{9}$$

where the product is to be taken over all primes p.

(iii) *The power series $\zeta^*_{\Lambda,p}(s)$ are rational functions in p^{-s}. That is, for each prime p there are polynomials $Z^*_p,\ N^*_p \in \mathbb{Z}[x]$ such that $\zeta^*_{\Lambda,p}(s) = Z^*_p(p^{-s})/N^*_p(p^{-s})$ holds. The polynomials $Z^*_p,\ N^*_p$ can be chosen to have bounded degree as p varies.*

An explicit determination of the local Euler factors (that is of the polynomials $Z^*_p,\ N^*_p$) of the zeta functions has been carried out in many cases including infinite families of examples. The methods range from ingenious elementary arguments to the use of algebraic geometry (resolving singularities). In several cases computer assistance was used (see [51]). See Section 6 for a selection of these examples. The database [52] collects comprehensive information on many examples treated so far. In very few cases the zeta function could be described by a closed formula in terms of the Riemann zeta function like in (6) or (7).

Note that, as a consequence of Theorem 1.2, the series (2), (3) and (5) converge to holomorphic functions in some right half-plane of $\mathbb{C}$. In fact the coefficients in (2), (3) and (5) are non-negative, hence by a well known theorem of E. Landau there is $\alpha \in \mathbb{R} \cup \{-\infty\}$ such that the series in question converges (absolutely and locally uniformly) for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \alpha$ and diverges if $\mathrm{Re}(s) < \alpha$. This α is called the *abscissa of convergence* of the series.

We wish to report the following theorem which collects together the main results of [13].

Theorem 1.3. *For $\Lambda \in \mathcal{T}$ the following hold.*

(i) *The abscissa of convergence α^*_Λ of $\zeta^*_\Lambda(s)$ is a rational number.*

(ii) *There is a $\delta > 0$ such that $\zeta^*_\Lambda(s)$ can be meromorphically continued to the region $\{\, s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha^*_\Lambda - \delta \,\}$.*

(iii) *The line $\{\, s \in \mathbb{C} \mid \mathrm{Re}(s) = \alpha^*_\Lambda \,\}$ contains at most one pole of $\zeta^*_\Lambda(s)$ (at the point $s = \alpha^*_\Lambda$).*

We define b^*_Λ to be the order of the pole of $\zeta^*_\Lambda(s)$ in $s = \alpha^*_\Lambda$. Using another theorem of E. Landau we find $b^*_\Lambda \geq 1$. Theorem 1.3 has as an immediate consequence:

Corollary 1.1. *Let Λ be in $\mathcal{T}$. Let*

$$s^*_\Lambda(N) := \sum_{n=1}^{N} a^*_\Lambda(n) \quad (N \in \mathbb{N}) \tag{10}$$

be the summatory function of the counting function a^*_Λ. We have

$$s^*_\Lambda(N) \sim c^*_\Lambda N^{\alpha^*_\Lambda} \log(N)^{b^*_\Lambda - 1} \tag{11}$$

with $c^*_\Lambda \in \mathbb{R}$.

The formula (11) means that the right hand side divided by the left hand side tends to 1 as N tends to infinity. The corollary follows from Theorem 1.3 using Tauber theory (see [38]). Note that the third property of the zeta function is essential for this application. Note also that c^*_Λ is equal to the lowest coefficient of the Laurent series representing ζ^*_Λ near the pole in $s = \alpha^*_\Lambda$.

Having defined the new invariants $\alpha^*_\Lambda \in \mathbb{Q}$, $b^*_\Lambda \in \mathbb{N}$ and $c^*_\Lambda \in \mathbb{R}$ for every Λ in $\mathcal{T}$ we are lead to

Problem 1.1. Relate $\alpha^*_\Lambda, b^*_\Lambda, c^*_\Lambda \in \mathbb{R}$ to structural properties of Λ.

This problem is solved when Λ is the ring of integers of a number field for the ideal zeta function. In this case $\alpha^\triangleleft_\Lambda = b^\triangleleft_\Lambda = 1$ and the value of $c^\triangleleft_\Lambda$ is given by Dirichlet's class number formula (see [32]). In the general case we have only the very scarce information reported in later sections. The following asymptotic relations can be read off from formulas (6) and (7), they reveal the values of our invariants.

$$s^{<}_{H_3}(N) \sim \frac{\zeta(2)^2}{2\zeta(3)} N^2 \log(N), \quad s^{<}_{sl_2(\mathbb{Z})}(N) \sim \frac{20\zeta(2)^2\zeta(3)}{31\zeta(5)} N^2 \tag{12}$$

The examples above illustrate that α^*_Λ can often be any natural number. However examples described in Section 6 show that $5/2$ and $7/2$ are also possible values of α^*_Λ. Considering $\mathbb{Z}^d$ ($d \in \mathbb{N}$) as a ring we have $\zeta^\triangleleft_{\mathbb{Z}^d}(s) = \zeta(s)^d$ and hence $b^\triangleleft_{\mathbb{Z}^d} = d$. Examples from Section 6 illustrate that b^*_Λ can take the values 1, 2, 3, 4. In fact, the Heisenberg group H_3 has $b^{<}_{H_3} = 2$. This is most of the knowledge we so far have on

Problem 1.2. What is the range of the pairs $(\alpha^*_\Lambda, b^*_\Lambda)$ as Λ varies over $\mathcal{T}$?

Problem 1.2 has many more concrete variants, let us mention one of them. Define $\mathbf{S}^*_{\text{group}} := \{\alpha^*_G\} \subset \mathbb{R}$ to be the set of abscissas of convergence of the subgroup or normal subgroup zeta functions as G varies over all finitely generated torsion-free nilpotent groups. Define $\mathbf{S}^*_{\text{ring}} := \{\alpha^*_R\} \subset \mathbb{R}$ similarly as R varies over all rings in $\mathcal{T}$. Let us briefly explain the proof of

Proposition 1.1. *The set $\mathbf{S}^{<}_{\text{group}} \subset \mathbb{R}$ is discrete, that is below any real number there are only finitely many members of $\mathbf{S}^{<}_{\text{group}}$.*

Let G be a torsion-free nilpotent group and G^{ab} its abelianisation. Let $h(G)$ be the Hirsch-length of G, that is the maximal number of infinite cyclic factors appearing in a composition series of G. A simple argument shows $h(G^{\text{ab}}) \leq \alpha^{*}_{G} \leq h(G)$. These two inequalities have been improved in several directions (see [27] and [42]). An unpublished result of D. Segal gives the lower bound

$$\bigl(3 - \sqrt{2}\bigr)h(G) - \frac{1}{2} \leq \alpha^{<}_{G}. \tag{13}$$

The main result of [4] implies that once we fix the Hirsch-length of G there is a universal denominator which all denominators of local zeta functions of nilpotent groups of that Hirsch-length have to divide. The results of Sections 2 and 3 together with (13) prove Proposition 1.1. The results of [4] apply also to the normal subgroup zeta function and to the zeta functions of rings, but a replacement for (13) has not been found. So we raise

Problem 1.3. Are the sets $\mathbf{S}^{\triangleleft}_{\text{group}}$, $\mathbf{S}^{<}_{\text{ring}}$ and $\mathbf{S}^{\triangleleft}_{\text{ring}}$ discrete? If not, what are their accumulation points?

Are $\Lambda_1, \Lambda_2 \in \mathcal{T}$ isomorphic if their zeta functions agree? Questions of this nature are traditionally called isospectrality problems. Examples of non-isomorphic rings of integers R_1, R_2 in number fields with $\zeta^{\triangleleft}_{R_1}(s) = \zeta^{\triangleleft}_{R_2}(s)$ are contained in [44]. The two finitely generated nilpotent groups (of class 2) G_1, G_2 described in Example 4 of Section 6 are not isomorphic but have isomorphic profinite completions. Hence $\zeta^{<}_{G_1}(s) = \zeta^{<}_{G_2}(s)$, $\zeta^{\triangleleft}_{G_1}(s) = \zeta^{\triangleleft}_{G_2}(s)$ both hold. These examples show that the isospectrality problem in general has a negative answer. But there remains:

Problem 1.4. Suppose that $\zeta^{*}_{\Lambda_1}(s) = \zeta^{*}_{\Lambda_2}(s)$ holds for both or at least one of the possibilities $* \in \{<, \triangleleft\}$ for $\Lambda_1, \Lambda_2 \in \mathcal{T}$. Which structural invariants of Λ_1 and Λ_2 are the same? For example, are the profinite completions of Λ_1 and Λ_2 isomorphic?

For a more extensive discussion of isospectrality problems see [12]. This paper also contains an example of a group G which satisfies $\zeta^{<}_{G}(s) = \zeta^{<}_{\mathbb{Z}^2}$ but which does not have the same profinite completion as $\mathbb{Z}^2$. The group G is one of the plane crystallographic groups, it has $\mathbb{Z}^2$ as a subgroup of index 2 but it is not nilpotent.

In this survey we mainly discuss properties of the zeta functions of groups and rings. There are many topics not treated here, see [20] and [10] for relations to other subjects. Connections to the by now vast field of subgroup growth are not treated here. For this see the surveys [34] and [35].

In Sections 2, 3 we describe the proofs of Theorems 1.2 and 1.3. As a first step the Euler factors $\zeta^{*}_{\Lambda,p}(s)$ are described as certain p-adic integrals. These are evaluated by the methods of p-adic integration. Having obtained explicit formulas we multiply the (global) Euler product by an Artin L-function to enlarge its region of convergence. Section 4 discusses the variation with p of the Euler factors. Certain functional equations of the Euler factors are the subject of Section 5. Section 6 contains examples. In Section 7 we describe variations on the zeta function theme.

2. p-adic formalism

While the proof of the first two items of Theorem 1.2 is elementary, the third requires an expression for the local Euler factor $\zeta^*_{\Lambda,p}(s)$ (p a prime) of the zeta function $\zeta^*_\Lambda(s)$ in terms of a certain p-adic integral. We shall briefly explain this procedure in the case when $\Lambda = R \in \mathcal{T}$ is a ring and $* = \triangleleft$. For more details see [27] Section 3 or [13].

Let p be a prime. We write $\mathbb{Q}_p$ for the field of p-adic numbers and $\mathbb{Z}_p$ for its ring of integers. For $x \in \mathbb{Q}_p$ we define $v_p(x)$ to be the p-adic valuation of x and $|x|_p$ to be the normalised p-adic absolute value. We write $\mathrm{Tr}_d(\mathbb{Z})$, $\mathrm{Tr}_d(\mathbb{Z}_p)$ ($d \in \mathbb{N}$) for the space of upper triangular $d \times d$-matrices with entries in $\mathbb{Z}$ respectively $\mathbb{Z}_p$. We think of $\mathrm{Tr}_d(\mathbb{Z}_p)$ being identified with $\mathbb{Z}_p^{d(d+1)/2}$.

Let $R \in \mathcal{T}$ be a ring (with additive group isomorphic to $\mathbb{Z}^d$) and p a prime. We fix a $\mathbb{Z}$-basis of R. Analysing the conditions for the rows of an upper triangular matrix (in $\mathrm{Tr}_d(\mathbb{Z})$) to be a triangular basis of an ideal in R, we find polynomials

$$f_1,\ g_1, \dots, f_l, \quad g_l \in \mathbb{Z}[x_{11}, \dots, x_{dd}] \tag{14}$$

such that

$$\mathcal{M}^{\triangleleft}(R) := \{\, x \in \mathrm{Tr}_d(\mathbb{Z}) \mid f_1(x) \mid g_1(x), \dots, f_l(x) \mid g_l(x) \,\} \tag{15}$$

is exactly the set of upper triangular matrices with entries in $\mathbb{Z}$ for which the rows generate an ideal in R. Here we write $a \mid b$ if the integer a divides the integer b. We now use our $\mathbb{Z}$-basis of R also as a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p \otimes_{\mathbb{Z}} R$. We conclude that

$$\mathcal{M}^{\triangleleft}(R,p) := \{\, x \in \mathrm{Tr}_d(\mathbb{Z}_p) \mid v_p(f_i(x)) \le v_p(g_i(x)) \text{ for } i = 1, \dots, l \,\} \tag{16}$$

is exactly the set of upper triangular matrices with entries in $\mathbb{Z}_p$ for which the rows additively generate an ideal in $\mathbb{Z}_p \otimes_{\mathbb{Z}} R$.

The map $\mathfrak{a} \to \mathfrak{a} \cap R$ sets up a one to one correspondence between the ideals of index p^n ($n \in \mathbb{N}$) in $\mathbb{Z}_p \otimes_{\mathbb{Z}} R$ and ideals of the same index in R. An exercise in p-adic integration shows that

$$\zeta^{\triangleleft}_{R,p}(s) = (1 - p^{-1})^{-d} \int_{\mathcal{M}^{\triangleleft}(R,p)} |x_{11}|_p^{s-n} |x_{22}|_p^{s-n+1} \dots |x_{dd}|_p^{s-1}\, dx \tag{17}$$

holds for every prime p with dx the normalised Haar measure on $\mathrm{Tr}_d(\mathbb{Z}_p) = \mathbb{Z}_p^{d(d+1)/2}$. The same approach applies to the subring zeta function of a ring $R \in \mathcal{T}$. See Section 3 of [27] or Section 5 of [13] for more details.

A similar, slightly more elaborate, analysis yields polynomials (14) in $d(d+1)/2$ variables such that formula (17) holds in case $\Lambda = G$ is a finitely generated torsion-free nilpotent group. For more details see Section 2 of [27] and Section 5 of [13]. Here, due to the use of Lie ring methods, finitely many primes have to be excluded. The natural number d has to be taken to be the Hirsch-length of G.

The proof in [27] of the rationality of these p-adic integrals relied on observing that $\mathcal{M}^*(\Lambda, p)$ are definable subsets in the language of fields. One can then apply a theorem of Denef [1] which establishes the rationality of definable p-adic integrals. Denef's proof relies on an application of Macintyre's quantifier elimination for the theory of $\mathbb{Q}_p$ which simplifies in a generally mysterious way the description of definable subsets like $\mathcal{M}^*(\Lambda, p)$. In the next section we shall report on a concrete formula computing integrals like (17) which replaces the use of the model theoretic black box in the proof of the rationality.

3. *p*-adic and adelic cone integrals

We define here certain Euler products with factors given by p-adic integrals which are generalized versions of the p-adic integrals occurring in formula (17). We then analyze the analytical properties of these Euler products.

Let m be a natural number. A collection of polynomials

$$\mathcal{D} = (f_0, g_0; f_1, g_1, \dots, f_l, g_l) \quad (f_0, g_0, f_1, g_1, \dots, f_l, g_l \in \mathbb{Q}[x_1, \dots, x_m]) \tag{18}$$

is called cone integral data. We associate to $\mathcal{D}$ the following closed subset of $\mathbb{Z}_p^m$ (p a prime)

$$\mathcal{M}(\mathcal{D}, p) := \{ x \in \mathbb{Z}_p^m \mid v_p(f_i(x)) \le v_p(g_i(x)) \text{ for } i = 1, \dots, l \} \tag{19}$$

and a p-adic integral with conventions as in Section 2:

$$Z_{\mathcal{D}}(s, p) = \int_{\mathcal{M}(\mathcal{D}, p)} |f_0(x)|_p^s \, |g_0(x)|_p \, dx. \tag{20}$$

Note that Section 2 shows that the local zeta functions of the $\Lambda \in \mathcal{T}$ are special cases of the $Z_{\mathcal{D}}(s, p)$. The p-adic integral (20) is easily seen to exist for $s \in \mathbb{C}$ with sufficiently large real part. It can be expressed as a power series $Z_{\mathcal{D}}(s) = \sum_{i=0}^{\infty} a_{p,i}\, p^{-is}$ with non-negative integer coefficients and $a_{p,0} \neq 0$. In fact, a result of Denef [1] says that the power series in (20) is rational in p^{-s}. Given the cone integral data $\mathcal{D}$ we can define $Z_{\mathcal{D}}(s, p)$ for every prime p. We use this to define an Euler product

$$Z_{\mathcal{D}}(s) = \prod_p (a_{p,0}^{-1} \cdot Z_{\mathcal{D}}(s, p)) \tag{21}$$

which we call the *global* or *adelic cone integral*. In fact, with appropriate normalisation of measures, $Z_{\mathcal{D}}(s)$ can be defined as an adelic integral (see [39] for a special case).

Special cases of the p-adic integrals (20) appear in [28]. We use an adaptation of a method to calculate p-adic integrals from [1] and [2] to show:

Proposition 3.1. *Let $\mathcal{D} = (f_0, g_0; f_1, g_1, \ldots, f_l, g_l)$ be cone integral data. Define the polynomial $F(x) := \prod_{i=0}^{l} f_i(x)g_i(x)$. Let (Y, h) be a resolution of singularities over $\mathbb{Q}$ of F. Let E_i ($i \in T$) be the irreducible components defined over $\mathbb{Q}$ of the reduced scheme $(h^{-1}(D))_{\mathrm{red}}$ where $D = \mathrm{Spec}(\mathbb{Q}[x]/F)$. Then the following hold.*

(i) *There exist rational functions $P_I(X, Y) \in \mathbb{Q}(X, Y)$ for each $I \subset T$ with the property that for almost all p:*

$$Z_{\mathcal{D}}(s) = \sum_{I \subset T} c_{p,I}\, P_I(p, p^{-s}) \tag{22}$$

where $c_{p,I} = |\{a \in \bar{Y}(\mathbb{F}_p) \mid a \in E_i \text{ if and only if } i \in I\}|$ and $\bar{Y}$ is the reduction mod p of the scheme Y.

(ii) *There is a closed polyhedral cone $\mathfrak{C} \subset \mathbb{R}^t_{\geq 0}$ where $t = |T|$ and a decomposition of $\mathfrak{C}$ into open simplicial pieces which we denote by R_k ($k \in \{0, 1, \ldots, w\}$). We arrange that $R_0 = (0, \ldots, 0)$ and $R_1, \ldots, R_q$ are the one-dimensional pieces. For each $k \in \{0, 1, \ldots, w\}$ let $M_k \subset \{1, \ldots, q\}$ denote the one-dimensional pieces in the closure of R_k. Then there are positive integers A_j, B_j for $j \in \{1, \ldots, q\}$ such that for almost all primes p:*

$$Z_{\mathcal{D}}(s) = \sum_{k=0}^{w} (p-1)^{I_k}\, p^{-m}\, c_{p,I_k} \prod_{j \in M_k} \frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}} \tag{23}$$

where I_k is the subset of T defined so that $i \in T \backslash I_k$ if and only if the i-th coordinate is zero for all elements of R_k.

The study of p-adic integrals like (20) has been initiated by J. Igusa. His fundamental results are documented in [28]. The references in [28] provide access to the vast literature on this subject. Previous to the results documented here the global or adelic versions (21) have only received attention in special cases (see [39]). Using various methods from analytic number theory and arithmetic geometry we show in [13] that Proposition 3.1 implies:

Corollary 3.1. *Let $\mathcal{D} = (f_0, g_0; f_1, g_1, \ldots, f_l, g_l)$ be cone integral data. Suppose $Z_{\mathcal{D}}(s)$ is not the constant function.*

(i) *The abscissa of convergence $\alpha = \alpha_{\mathcal{D}}$ of $Z_{\mathcal{D}}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a rational number.*

(ii) *$Z_{\mathcal{D}}(s)$ has a meromorphic continuation to* $\mathrm{Re}(s) > \alpha - \delta$ *for some $\delta > 0$.*

(iii) *The line $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) = \alpha_\Lambda \}$ contains only one pole of $\zeta_{\mathcal{D}}(s)$ at $s = \alpha_\Lambda$.*

In fact, we multiply $Z_{\mathcal{D}}(s)$ by the Artin L-function corresponding to the permutation representation of the absolute Galois group of $\mathbb{Q}$ on the irreducible components of the E_i ($i \in T$) appearing in Proposition 3.1. Using the estimates of Hasse and

Weil for the number of points on algebraic varieties over finite fields on the c_{p,I_k} in formula (23) we can analyse the analytic properties of the product of $Z_{\mathcal{D}}(s)$ with its Artin L-function near the abscissa of convergence of $Z_{\mathcal{D}}(s)$.

Theorem 1.3 is a consequence of this corollary together with the discussion in Section 2.

Following classical analytic number theory it is natural to ask how far the adelic cone integrals $Z_{\mathcal{D}}(s)$ can be meromorphically continued to the left. The analysis of special cases shows that often natural boundaries arise as one continues to the left (see [11]). That is, in these cases, poles or zeroes of the continued function accumulate densely to the points of a vertical line in $\mathbb{C}$. Beyond this line no continuation is possible. The following problems seem to be of interest.

Problem 3.1. Find all cone integral data $\mathcal{D}$ such that $\zeta_{\mathcal{D}}(s)$ has a meromorphic continuation to $\mathbb{C}$, or at least give sufficient conditions for this to happen.

Problem 3.2. Find all $\Lambda \in \mathcal{T}$ such that $\zeta_\Lambda(s)$ has a meromorphic continuation to $\mathbb{C}$, or at least give sufficient conditions for this to happen.

Problem 3.3. Show that either $\zeta_{\mathcal{D}}(s)$ has a meromorphic continuation to $\mathbb{C}$ or that there is some rational number $\beta_{\mathcal{D}}$ such that the line $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) = \beta_\Lambda \}$ is a natural boundary.

In [14] and [16] we attempt, partially successfully, to replace the zeta function by a ghost zeta function which has more amenable analytic properties but which has Euler factors which are in a specific sense near to those of the original zeta function.

The process of continuation to the left ties up the Dirichlet series $\zeta_{\mathcal{D}}(s)$ with the zeta functions defined by A. Weil and R. Langlands for smooth $\mathbb{Q}$-defined projective algebraic varieties. Let us report on a special example. Let $y^2-x^3-ax-b\,(a,\ b \in \mathbb{Q})$ be a polynomial representing an elliptic curve E. Define

$$Z_{E,p}(s) := \int_{\mathbb{Z}_p^2} |y^2 - x^3 - ax - b|_p^s \, dx, \quad Z_E(s) := \prod_p (\lambda_p^{-1} Z_{E,p}(s)) \qquad (24)$$

with appropriate normalisation factors λ_p. We have shown in [17] that the Dirichlet series $Z_E(s)$ converges for $\mathrm{Re}(s) > 0$. Moreover when attempting to continue $Z_E(s)$ to the left, the symmetric power L-functions attached to E arise. It is conjectured that these symmetric power L-functions can all be meromorphically continued to $\mathbb{C}$.[1] If this is true then $Z_E(s)$ can be meromorphically continued to the region $\mathrm{Re}(s) > -3/2$. Results of J. P. Serre concerning the Sato–Tate conjecture for E then imply that the line $\mathrm{Re}(s) = -3/2$ is a natural boundary beyond which no continuation is possible.

[1] Note added in proof: these conjectures have recently been proved.

4. The local factors: variation with *p*

The behaviour of the local factors as we vary p is one of the other major problems in the field. If we consider formula (6) we easily see that

$$\zeta_{H_3,p}(s) = \frac{W_1(p, p^{-s})}{W_2(p, p^{-s})} \tag{25}$$

where W_1, W_2 can be given without reference to the prime p as polynomials $W_1(X, Y) = (1 - Y)(1 - XY)(1 - X^2Y^2)(1 - X^3Y^2)$, $W_2(X, Y) = 1 - X^3Y^3$. Groups with this property are said to have *uniform subgroup* or *normal subgroup growth*. In [27] it is proved that a finitely generated free nilpotent group of class 2 has both uniform subgroup and normal subgroup growth. As revealed in [7] the following problem ties up intimately with classification problems of finite p-groups, in particular with Higman's PORC-conjecture.

Problem 4.1. Show that every finitely generated free nilpotent group has both uniform subgroup and normal subgroup growth.

We can also consider a similar variety of problems for rings. We define *uniform subring* or *ideal growth* as above in the group case and raise

Problem 4.2. Show that the following Lie rings have uniform subring and ideal growth.

- Free nilpotent Lie rings of finite $\mathbb{Z}$-rank,
- $sl_n(\mathbb{Z})$ ($n \in \mathbb{N}$) or any other integer version of a simple Lie algebra over $\mathbb{C}$.

Let us now consider the Heisenberg group H_3 with entries from the ring of integers of a quadratic number field. The behaviour of the local factors of its zeta functions depends on how p behaves in the number field [27]. That is formulas like (25) hold, but finitely many pairs of polynomials are needed to describe the variation of the local factor of the zeta function with p. Groups with this property are said to have *finitely uniform subgroup* or *normal subgroup growth*. There is a similar concept in the case of rings.

For a long time this was the only type of variation with p which was known. Our explicit formula however takes the subject away from the behaviour of primes in number fields to the problem of counting points modulo p on a variety, a question which is in general wild and far from the uniformity predicted by all previous examples seen in [27]. Two papers [8] and [9] by the first author contain the following example of a class two nilpotent group of Hirsch length 9 whose zeta function depends on counting points mod p on the elliptic curve $y^2 = x^3 - x$. Define

$$G = \left\langle \begin{matrix} x_1, x_2, x_3, x_4, x_5, x_6, \\ y_1, y_2, y_3 \end{matrix} \;\middle|\; \begin{matrix} [x_1, x_4] = y_3, [x_1, x_5] = y_1, [x_1, x_6] = y_2, \\ [x_2, x_4] = y_2, [x_2, x_6] = y_1, [x_3, x_4] = y_1, \\ [x_3, x_5] = y_3 \end{matrix} \right\rangle \tag{26}$$

with the convention that commutators not mentioned are equal to 1. By [9] there exist rational functions $P_1(X, Y), P_2(X, Y) \in \mathbb{Q}(X, Y)$ such that for almost all primes p

$$\zeta^{\triangleleft}_{G,p}(s) = P_1(p, p^{-s}) + |E(\mathbb{F}_p)| P_2(p, p^{-s}), \tag{27}$$

where E is the elliptic curve $y^2 = x^3 - x$. In [9] this formula is used to show that G is not finitely uniform. To see where the elliptic curve is hidden in the above presentation, take the determinant of the 3×3 matrix (a_{ij}) with entries $a_{ij} = [x_i, x_{j+3}]$ and you will get the projective version of E.

Formula (23) shows that the variation type with p of the Euler factors $\zeta_{\mathcal{D},p}(s)$ ($\mathcal{D}$ cone condition data) is that of functions counting points on a $\mathbb{Q}$-defined algebraic variety modulo primes p. But might their be further restrictions once we consider the Euler factors $\zeta_{\Lambda,p}(s)$ for $\Lambda \in \mathcal{T}$?

Problem 4.3. Let V be a $\mathbb{Q}$-defined algebraic variety. Is there $\Lambda_V \in \mathcal{T}$ such that there are rational functions $P_1(X, Y), P_2(X, Y) \in \mathbb{Q}(X, Y)$ such that for almost all primes p

$$\zeta_{\Lambda_V,p}(s) = P_1(p, p^{-s}) + |V(\mathbb{F}_p)| P_2(p, p^{-s}) \tag{28}$$

holds?

The consideration of zeta functions obtained by motivic integration (see [18]) sheds some light on this new dialogue between groups and rings and questions of arithmetic geometry.

5. Functional equations of the local factors

There is another remarkable feature of many of the rational functions representing the local zeta function of nilpotent groups: they satisfy a certain palindromic symmetry. Let us explain this in the case of the normal subgroup zeta function of $F_{2,3}$, the free nilpotent group of class two on three generators. The group $F_{2,3}$ is torsion-free and has Hirsch-length 6. From [27] we know that

$$\zeta^{\triangleleft}_{F_{2,3},p}(s) = \left.\frac{1 + X^3Y^3 + X^4Y^3 + X^6Y^5 + X^7Y^5 + X^{10}Y^8}{(1-Y)(1-XY)(1-X^2Y)(1-X^8Y^5)(1-X^6Y^9)}\right|_{X=p,\, Y=p^{-s}} \tag{29}$$

holds for every prime p. Let us replace p by p^{-1} (and p^{-s} by p^s) in this expression. Indicating this replacement by $p \to p^{-1}$, we find:

$$\zeta^{\triangleleft}_{F_{2,3},p}(s)|_{p\to p^{-1}} = p^{15-9s} \zeta^{\triangleleft}_{F_{2,3},p}(s). \tag{30}$$

This phenomenon was found in all examples of all finitely generated nilpotent groups of class 2 and Lie rings of nilpotency class 2 where explicit computations have been done. We pose here the

Problem 5.1. Let G be a nilpotent group of class 2 and Hirsch-length h. Assume that the quotient of G modulo its center (which is abelian) has torsion-free rank m. Show that

$$\zeta^{<}_{G,p}(s)|_{p\to p^{-1}} = (-1)^d \, p^{\frac{h(h-1)}{2}-hs} \, \zeta^{<}_{G,p}(s), \tag{31}$$

$$\zeta^{\triangleleft}_{G,p}(s)|_{p\to p^{-1}} = (-1)^d \, p^{\frac{h(h-1)}{2}-(h+m)s} \, \zeta^{\triangleleft}_{G,p}(s) \tag{32}$$

hold for almost all primes p.

In [48] C. Voll is able to answer Problem 5.1 affirmatively for the special case of local zeta functions counting normal subgroups of torsion-free class 2 nilpotent groups which have a centre of $\mathbb{Z}$-rank 2 by giving explicit formulas for the local zeta functions in this case. C. Voll [49] and P. Paajanen [40], [43] and [41] have also confirmed the functional equation for the normal subgroup zeta function in more general settings by analysing the geometry of the Pfaffian hypersurface associated to presentations of class 2 nilpotent groups. Note however that the functional equation for zeta functions of nilpotent groups is not a completely general phenomenon. The Lie ring $\mathfrak{L}_W$ introduced in Example 3 of the next section has nilpotency class 3. The Euler factors of the ideal counting zeta function do not satisfy a functional equation, although the Euler factors of the subring zeta function do have such a symmetry.

Problem 5.1 should be seen in connection with a result of Denef and Meuser [2] who prove that the rational expression (in p^{-s}) corresponding to the Igusa-type p-adic integral

$$Z_{\{g_0,\,1\},p}(s) := \int_{\mathbb{Z}^m} |g_0(x)|_p^s \, dx \tag{33}$$

satisfy a functional equation if $g_0 \in \mathbb{Z}_p[x_1, \ldots, x_m]$ is absolutely irreducible and defines a smooth projective hypersurface over the finite field $\mathbb{F}_p$. A key role in their proof is played by the functional equation satisfied by the algebraic geometric zeta function for this hypersurface proved by A. Weil.

In [48] C. Voll uses the functional equation for the local zeta functions of elliptic curves to prove that the zeta function (27) of the nilpotent group encoding an elliptic curve in its presentation has a functional equation of the type predicted by (31). The paper [31] of B. Klopsch and C. Voll treats interesting new counting problems related to orthogonal and unitary groups over finite fields which arose in the study of functional equations.

The only counterexamples to the functional equations for zeta functions of groups and rings relate to counting normal subgroups in groups or ideals in rings. We therefore raise the following:

Problem 5.2. Let Λ be in $\mathcal{T}$. Show that there are rational numbers a, b and c such that

$$\zeta^{<}_{\Lambda,p}(s)|_{p\to p^{-1}} = (-1)^c \, p^{as+b} \, \zeta^{<}_{\Lambda,p}(s) \tag{34}$$

for almost all primes p.

In his thesis [51] L. Woodward analyses a general setting in which certain cone integrals are conjectured to have functional equations which could generalise the result of Denef and Meuser. The cone integral data has to satisfy what Woodward calls a homogeneity condition, namely $\deg(g_i) = \deg(f_i) + 1$ for $i = 1, \dots, l$. The cone integrals describing zeta functions counting all subgroups or subrings satisfy this homogeneity condition in contrast to the normal subgroup and ideal zeta functions. Using results of Stanley on functional equations of polyhedral cones Woodward can prove the functional equation in the special case that all the polynomials of the cone integral data are monomials.

6. Examples

This section contains a brief description of the information obtained so far on the zeta functions of several series of Lie rings. We also describe the pair of finitely generated nilpotent groups which solves the isospectrality problem negatively.

Example 1. Let $\mathfrak{F}(2, n)$ be the free nilpotent Lie ring of class two on $n \in \mathbb{N}$ $(n \geq 2)$ generators. This Lie ring has $\mathbb{Z}$-rank $h(n) = n + n(n+1)/2$. Note that the Lie ring of the Heisenberg group H_3 is isomorphic to $\mathfrak{F}(2, 2)$. An explicit formula for the Euler factors of the ideal zeta function is given by C. Voll in [48] (see [40] for special cases). From these

$$\alpha^{\triangleleft}_{\mathfrak{F}(2,n)} = \max\left\{ n, \frac{\left(\frac{n(n-1)}{2} - j\right)(n+j)+1}{h(n) - j} \;\middle|\; j = 1, \dots, \frac{n(n-1)}{2} - 1 \right\} \tag{35}$$

can be deduced. This formula shows that for $n \geq 5$, the abscissa of convergence of the global ideal zeta function is greater than n and is usually not an integer. However, sometimes it may just happen to be an integer. The only n in the range $5 \leq n \leq 200$ for which this happens is $n = 26$. Furthermore the ideal growth of $\mathfrak{F}(2, n)$ is uniform and the Euler factors $\zeta_{\mathfrak{F}(2,n),p}$ satisfy the functional equation of Problem 5.1 (see [50]).

Example 2. Let n be a natural number with $n \geq 2$. Define $\mathfrak{G}(n)$ to be the Lie ring

$$\mathfrak{G}(n) := \langle\, z,\ x_1, \dots, x_{n-1},\ y_1, \dots, y_{n-1} \mid [z, x_i] = y_i\ \ (i = 1, \dots, n-1)\,\rangle. \tag{36}$$

Our convention again is that all commutators between the generators not mentioned are equal to 0. Hence $\mathfrak{G}(n)$ has nilpotency class two and $\mathbb{Z}$-rank $2n - 1$. D. Grenham [22] has determined explicit formulas for the ideal zeta functions of $\mathfrak{G}(n)$ for $n = 2, 3, 4, 5$. Let us report his formula in case $n = 4$. Define $W_4(X, Y)$ to be the rational function

$$W_4(X, Y) := \frac{1 + X^4Y^3 + X^5Y^3 + X^8Y^5 + X^9Y^5 + X^{18}Y^8}{(1-Y)(1-XY)(1-X^2Y)(1-X^3Y)(1-X^6Y^3)(1-X^{10}Y^5)}. \tag{37}$$

Grenham's formula reads as

$$\zeta^{\triangleleft}_{\mathfrak{G}(4),p}(s) = W_4(X,Y)|_{X=p,\,Y=p^{-s}}. \tag{38}$$

From this it is immediately clear that $\mathfrak{G}(4)$ has uniform ideal growth. Also $\alpha^{\triangleleft}_{\mathfrak{G}(4)} = 4$ and $b^{\triangleleft}_{\mathfrak{G}(4)} = 1$ can be read off. Further analysis of the numerator in (37) shows that the global zeta function $\zeta^{\triangleleft}_{\mathfrak{G}(4)}(s)$ has a natural boundary at $\mathrm{Re}(s) = 9/5$ (see [11]).

Using methods of algebraic geometry C. Voll [49] has developed a closed formula for $\zeta^{\triangleleft}_{\mathfrak{G}(n),p}(s)$ which holds for every $n \geq 2$ and every prime p. This formula shows that $\mathfrak{G}(n)$ has uniform ideal growth for every $n \geq 2$, it also confirms the conjectures from Section 5 concerning functional equations of the Euler factors. Also, for $n \geq 6$ the abscissa of convergence $\alpha^{\triangleleft}_{\mathfrak{G}(n)}$ is greater than n, and it is in general not an integer. Indeed, if $6 \leq n \leq 200$, the abscissa of convergence is an integer if and only if $n = 2N^2 + 6N + 5$ for some integer N.

D. Grenham [22] has also studied the subring zeta function of $\mathfrak{G}(n)$. We cite from [22] the following pole orders:

$$b^{<}_{\mathfrak{G}(3)} = 2, \quad b^{<}_{\mathfrak{G}(4)} = 2, \quad b^{<}_{\mathfrak{G}(5)} = 3. \tag{39}$$

The corresponding abscissas of convergence are

$$\alpha^{<}_{\mathfrak{G}(3)} = 3, \quad \alpha^{<}_{\mathfrak{G}(4)} = 4, \quad \alpha^{<}_{\mathfrak{G}(5)} = 5. \tag{40}$$

Example 3. The following example of a Lie ring played an important role in the development of the conjectures from Section 5 concerning functional equations of the Euler factors. Define

$$\mathfrak{L}_W := \langle\, z,\ w_1,\ w_2,\ x_1,\ x_2,\ y \mid [z,w_1] = x_1,\ [z,w_2] = x_2,\ [z,x_1] = y \,\rangle. \tag{41}$$

This Lie ring has nilpotency class 3 and $\mathbb{Z}$-rank 6. It was discovered and extensively studied by L. Woodward in [51]. The Lie ring $\mathfrak{L}_W$ has uniform subring and ideal growth but only the local subring counting zeta function satisfies a functional equation. We further report from [51]:

$$\alpha^{<}_{\mathfrak{L}_W} = 3,\ b^{<}_{\mathfrak{L}_W} = 4, \quad \alpha^{\triangleleft}_{\mathfrak{L}_W} = 3,\ b^{\triangleleft}_{\mathfrak{L}_W} = 1. \tag{42}$$

The global zeta function $\zeta^{<}_{\mathfrak{L}_W}(s)$ has a natural boundary at $\mathrm{Re}(s) = 17/7$ whereas $\zeta^{\triangleleft}_{\mathfrak{L}_W}(s)$ has a natural boundary at $\mathrm{Re}(s) = 7/6$.

Example 4. In [23] it is proved that the following two finitely generated nilpotent groups

$$G_1 = \left\langle \begin{matrix} g_1, g_2, g_3, g_4, \\ z_1,\ z_2 \end{matrix} \;\middle|\; \begin{matrix} [g_1,g_2] = 1,\ [g_3,g_4] = 1,\ [g_1,g_3] = z_1, \\ [g_1,g_4] = z_2,\ [g_2,g_3] = z_2,\ [g_2,g_4] = z_1^{-5} \end{matrix} \right\rangle, \tag{43}$$

$$G_2 = \left\langle \begin{matrix} g_1, g_2, g_3, g_4, \\ z_1,\ z_2 \end{matrix} \;\middle|\; \begin{matrix} [g_1,g_2] = 1,\ [g_3,g_4] = 1,\ [g_1,g_3] = z_1, \\ [g_1,g_4] = z_2,\ [g_2,g_3] = z_1^{-1}z_2^{2},\ [g_2,g_4] = z_1^{-3}z_2 \end{matrix} \right\rangle \tag{44}$$

have the same profinite completion but are not isomorphic. It follows that both their zeta functions are the same. Both groups have Hirsch-length equal to 6 and are of nilpotency class 2. These groups come as special cases of an infinite series of l-tuples ($l \geq 2$) of examples of such groups arising from a number theoretic setting.

7. Variation

We have put the emphasis on counting subgroups or normal subgroups in nilpotent groups and on counting subrings or ideals in rings, however our results extend in a number of other directions.

(1) Variants of our zeta functions have been considered which count only subgroups with some added feature, for example characteristic subgroups or subgroups of a finitely generated torsion-free nilpotent group G which are isomorphic to G. Theorems 1.2 and 1.3 hold in this case and for many of these variants. In fact, there is always a p-adic formalism like in Section 2 which reduces Theorem 1.3 to Corollary 3.1 (see [27]). The paper [19] relates the zeta functions counting subgroups of G which are isomorphic to G to zeta functions defined by A. Weil for $\mathbb{Q}$-defined linear algebraic groups.

(2) The rationality result of Theorem 1.2 also holds for finitely generated nilpotent groups which are not necessarily torsion-free. In fact, the first author proved in [3] that this result extends to all finitely generated soluble groups of finite rank.

(3) In [12] it is proved that all crystallographic groups or more generally all finitely generated groups which contain an abelian subgroup of finite index have zeta functions which have a meromorphic continuation to all of $\mathbb{C}$. This is done by relating these zeta functions to zeta functions of orders in central simple $\mathbb{Q}$-algebras.

(4) The local zeta functions of the classical groups (see [14], [13]) can be expressed as p-adic cone integrals and our results apply to the corresponding Euler product.

(5) Let $g(n, c, d)$ be the number of finite nilpotent groups of size n, of nilpotency class bounded by c and generated by at most d elements. In [7] the zeta function

$$\zeta_{\mathcal{N}(c,d)}(s) := \sum_{n=1}^{\infty} g(n, c, d) n^{-s} \tag{45}$$

is shown to be expressible as the Euler product of p-adic cone integrals. Hence our results apply and give asymptotic results for the partial sums of the $g(n, c, d)$. The formalism of zeta functions has been applied successfully in [7] to solve conjecture **P**, which had appeared in connection with periodicity in trees connected with the classification problem for finite p-groups in terms of coclass.

(6) Thinking of Hilbert's basis theorem we might expect a connection between the ideal counting zeta function of a ring R and that of the polynomial ring $R[x]$ over R. This expectation is confirmed by a beautiful formula of D. Segal [45] which holds for Dedekind rings R.

(7) The formalism of zeta functions has been used to count representations of arithmetic and p-adic analytic groups in the papers [37] of B. Martin and A. Lubotzky, [30] of A. Jaikin-Zapirain and [33] of M. Larsen and A. Lubotzky.

References

[1] Denef, J., The rationality of the Poincaré series associated to the p-adic points on a variety. *Invent. Math.* **77** (1984), 1–23.

[2] Denef, J., Meuser, D., A functional equation of Igusa's local zeta function. *Amer. J. Math.* **113** (1991), 1135–1152.

[3] du Sautoy, M. P. F., Finitely generated groups, p-adic analytic groups and Poincaré series. *Ann. of Math.* **137** (1993), 639–670.

[4] du Sautoy, M. P. F., Zeta functions of groups and Lie algebras: uniformity. *Israel J. Math.* **86** (1994), 1–23.

[5] du Sautoy, M. P. F., The zeta function of $sl_2(\mathbb{Z})$. *Forum Math.* **12** (2000), 197–221.

[6] du Sautoy, M. P. F., Addendum to the paper: The zeta function of $sl_2(\mathbb{Z})$. *Forum Math.* **12** (2000), 383.

[7] du Sautoy, M. P. F., Counting p-groups and nilpotent groups. *Inst. Hautes Études Sci. Publ. Math.* **92** (2000), 63–112.

[8] du Sautoy, M. P. F., A nilpotent group and its elliptic curve: non-uniformity of local zeta functions of groups. *Israel J. Math.* **126** (2001), 269–288.

[9] du Sautoy, M. P. F., Counting subgroups in nilpotent groups and points on elliptic curves. *J. Reine Angew. Math.* **549** (2002), 1–21.

[10] du Sautoy, M. P. F., Zeta functions of groups: the quest for order versus the flight from ennui. *Groups St. Andrews 2001 in Oxford*, Vol. I, , London Math. Soc. Lecture Note Ser. 304, Cambridge University Press, Cambridge 2003, 150–189.

[11] du Sautoy, M. P. F., Natural boundaries for zeta functions of groups. Preprint.

[12] du Sautoy, M. P. F., McDermott, J. J., Smith, G. C., Zeta functions of crystallographic groups and meromorphic continuation. *Proc. London Math. Soc.* **79** (1999), 511–534.

[13] du Sautoy, M. P. F., Grunewald, F., Analytic properties of zeta functions and subgroup growth. *Ann. of Math.* **152** (2000), 793–833.

[14] du Sautoy, M. P. F., Grunewald, F., Zeta functions of classical groups and their friendly ghosts. *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), 1–6.

[15] du Sautoy, M. P. F., Grunewald, F., Analytic properties of Euler products of Igusa type zeta functions and subgroup growth of nilpotent groups. *C. R. Acad. Sci. Paris Sér. I Math.* **329** (1999), 351–356.

[16] du Sautoy, M. P. F., Grunewald, F., Zeta functions of groups: Zeros and friendly ghosts. *Amer. J. Math.* **124** (2002), 1–48.

[17] du Sautoy, M. P. F., Grunewald, F., Natural boundaries for Euler products of Igusa zeta functions of elliptic curves. Preprint.

[18] du Sautoy, M. P. F., Loeser, F., Motivic zeta functions of infinite dimensional Lie algebras. *Selecta Math.* **10** (2004), 253–303.

[19] du Sautoy, M. P. F., Lubotzky, A., Functional equations and uniformity for local zeta-functions of nilpotent groups. *Amer. J. Math.* **118** (1996), 39–90.

[20] du Sautoy, M. P. F., Segal, D., Zeta functions of groups. In *New Horizons in pro-p Groups* (ed. by M. P. F. du Sautoy, D. Segal, A. Shalev), Progr. Math. 184, Birkhäuser, Boston 2002, 249–286.

[21] du Sautoy, M. P. F., Taylor, G., The zeta function of sl_2 and resolution of singularities. *Math. Proc. Cambridge Philos. Soc.* **132** (2002), 57–73.

[22] Grenham, D., Some topics in nilotent group theory. DPhil Thesis, Oxford, 1988.

[23] Grunewald, F., Scharlau, R., A note on finitely generated torsion-free groups of class 2. *J. Algebra* **58** (1979), 162–175.

[24] Grunewald, F., Segal, D., Some general algorithms. II: Nilpotent Groups. *Ann. of Math.* **112** (1980), 585–617.

[25] Grunewald, F., Segal, D., Nilpotent groups of Hirsch length six. *Math. Z.* **179** (1982), 219–235.

[26] Grunewald, F., Segal, D., Reflections on the classification of torsion-free nilpotent groups. In *Group theory* (Essays for Philip Hall), ed. by K. W. Gruenberg and J. E. Roseblade, Academic Press, London 1984.

[27] Grunewald, F. J., Segal, D., Smith, G. C., Subgroups of finite index in nilpotent groups. *Invent. Math.* **93** (1988), 185–223.

[28] Igusa, J., *An introduction to the theory of local zeta functions*. AMS/IP Stud. Adv. Math. 14, Amer. Math. Soc., Providence, RI, International Press, Cambrigde, MA, 2002.

[29] Ilani, I., Zeta functions relating to the group $\mathrm{SL}_2(\mathbb{Z}_p)$. *Israel J. Math.* **109** (1999), 157–172.

[30] Jaikin-Zapirain, A., Zeta function of representations of compact p-adic analytic groups. *J. Amer. Math. Soc.* **19** (2006), 91–118.

[31] Klopsch, B., Voll, C., Igusa-type functions associated to finite formed spaces and their functional equations. Preprint, 2006.

[32] Lang, S., *Algebraic Number Theory*. Addison Wesley Publishing Company, Reading, MA, 1970.

[33] Larsen, M., Lubotzky, A., Counting representations: zeta functions and rates of growth. Preprint.

[34] Lubotzky, A., Counting finite index subgroups. In *Groups '93 Galway/St. Andrews*, Vol. 2, London Math. Soc. Lecture Note Ser. 212, Cambridge University Press, Cambridge 1995, 368–404.

[35] Lubotzky, A., Subgroup growth. In *Proceedings of the International Congress of Mathematicians* (Zürich, 1994), Vol. 1, Birkhäuser, Basel 1995, 309–317.

[36] Lubotzky, A., Mann, A., and Segal, D., Finitely generated groups of polynomial subgroup growth. *Israel J. Math.* **82** (1993) 363-371.

[37] Lubotzky, A., Martin, B., Polynomial representation growth and the congruence subgroup problem. *Israel J. Math.* **144** (2004), 293–316.

[38] Narkiewicz, W., *Number Theory*. World Scientific Publishing Co., Singapore 1983.

[39] Ono, T., An integral attached to a hypersurface. *Amer. J. Math.* **90** (1968), 1224–1236.

[40] Paajanen, P. M., The normal zeta function of the free class two nilpotent group on four generators. *Geom. Dedicata* **115** (2005), 135–163.

[41] Paajanen, P. M., Zeta functions of groups and arithmetic geometry. DPhil. Thesis, Oxford, 2005.

[42] Paajanen, P. M., On the degree of polynomial subgroup growth in class 2 nilpotent groups. *Israel J. Math.*, to appear.

[43] Paajanen, P. M., Zeta functions of groups and the Segre surface. Preprint.

[44] Perlis, R., On the equation $\zeta_K(s) = \zeta_{K'}(s)$. *J. Number Theory* **9** (1977), 342–360.

[45] Segal, D., Ideals of finite index in a polynomial ring. *Quart. J. Math. Oxford* **48** (1997), 83–92.

[46] Smith, G. C., Zeta functions of torsion free finitely generated nilpotent groups. Ph.D. thesis, Manchester (UMIST), 1983.

[47] Taylor, G., Zeta functions of algebras and resolution of singularities. Ph.D. thesis, Cambridge, 2001.

[48] Voll, C., Zeta functions of groups and enumeration in Bruhat-Tits buildings. *Amer. J. Math.* **126** (2004), 1005–1032.

[49] Voll, C., Functional equations for local normal zeta functions of nilpotent groups. *Geom. Funct. Anal.* **15** (2005), 274–295.

[50] Voll, C., Normal subgroup growth in free class 2 nilpotent groups. *Math. Ann.* **332** (2005), 67–79.

[51] Woodward, L., Zeta functions of groups: computer calculations and functional equations. DPhil Thesis, Oxford, 2005.

[52] Woodward, L., Zeta functions of Lie rings archive. http://www.lack-of.org.uk/zfarchive/.

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