Lecture Notes in Mathematics

Marcus du Sautoy
Luke Woodward

Zeta Functions of Groups and Rings

1925

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Springer

Lecture Notes in Mathematics 1925

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ISBN 978-3-540-74701-7 e-ISBN 978-3-540-74776-5

DOI 10.1007/978-3-540-74776-5

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2007936935

Mathematics Subject Classification (2000): 20E07, 11M41

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

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To our families

Preface

The study of the subgroup growth of infinite groups is an area of mathematical research that has grown rapidly since its inception at the Groups St. Andrews conference in 1985. It has become a rich theory requiring tools from and having applications to many areas of group theory. Indeed, much of this progress is chronicled by Lubotzky and Segal within their book [42].

However, one area within this study has grown explosively in the last few years. This is the study of the zeta functions of groups with polynomial subgroup growth, in particular for torsion-free finitely-generated nilpotent groups. These zeta functions were introduced in [32], and other key papers in the development of this subject include [10, 17], with [19, 23, 15] as well as [42] presenting surveys of the area.

The purpose of this book is to bring into print significant and as yet unpublished work from three areas of the theory of zeta functions of groups.

First, there are now numerous calculations of zeta functions of groups by doctoral students of the first author which are yet to be made into printed form outside their theses. These explicit calculations provide evidence in favour of conjectures, or indeed can form inspiration and evidence for new conjectures. We record these zeta functions in Chap. 2. In particular, we document the functional equations frequently satisfied by the local factors. Explaining this phenomenon is, according to the first author and Segal [23], "one of the most intriguing open problems in the area".

A significant discovery made by the second author was a group where all but perhaps finitely many of the local zeta functions counting normal subgroups do not possess such a functional equation. Prior to this discovery, it was expected that all zeta functions of groups should satisfy a functional equations. Prompted by this counterexample, the second author has outlined a conjecture which offers a substantial demystification of this phenomenon. This conjecture and its ramifications are discussed in Chap. 4.

Finally, it was announced in [16] that the zeta functions of algebraic groups of types B_l, C_l and D_l all possessed a natural boundary, but this work is also yet to be made into print. In Chap. 5 we present a theory of natural boundaries of two-variable polynomials. This is followed by Chap. 6 where the aforementioned result on the zeta functions of classical groups is proved, and Chap. 7, where we consider the natural boundaries of the zeta functions attached to nilpotent groups listed in Chap. 2.

The first author thanks Zeev Rudnick who first informed him of Conjecture 1.11, Roger Heath-Brown who started the ball rolling and Fritz Grunewald for discussions which helped bring the ball to a stop. The first author also thanks the Max-Planck Institute in Bonn for hospitality during the preparation of this work and the Royal Society for support in the form of a University Research Fellowship. The second author thanks the EPSRC for a Research Studentship and a Postdoctoral Research Fellowship, and the first author for supervision during his doctoral studies.

Oxford, *Marcus du Sautoy*
January 2007 *Luke Woodward*

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