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Luke Woodward

Zeta Functions of Groups and Rings

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$$\zeta_G(s)$$

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To our families

Preface

The study of the subgroup growth of infinite groups is an area of mathematical research that has grown rapidly since its inception at the Groups St. Andrews conference in 1985. It has become a rich theory requiring tools from and having applications to many areas of group theory. Indeed, much of this progress is chronicled by Lubotzky and Segal within their book [42].

However, one area within this study has grown explosively in the last few years. This is the study of the zeta functions of groups with polynomial subgroup growth, in particular for torsion-free finitely-generated nilpotent groups. These zeta functions were introduced in [32], and other key papers in the development of this subject include [10, 17], with [19, 23, 15] as well as [42] presenting surveys of the area.

The purpose of this book is to bring into print significant and as yet unpublished work from three areas of the theory of zeta functions of groups.

First, there are now numerous calculations of zeta functions of groups by doctoral students of the first author which are yet to be made into printed form outside their theses. These explicit calculations provide evidence in favour of conjectures, or indeed can form inspiration and evidence for new conjectures. We record these zeta functions in Chap. 2. In particular, we document the functional equations frequently satisfied by the local factors. Explaining this phenomenon is, according to the first author and Segal [23], “one of the most intriguing open problems in the area”.

A significant discovery made by the second author was a group where all but perhaps finitely many of the local zeta functions counting normal subgroups do not possess such a functional equation. Prior to this discovery, it was expected that all zeta functions of groups should satisfy a functional equations. Prompted by this counterexample, the second author has outlined a conjecture which offers a substantial demystification of this phenomenon. This conjecture and its ramifications are discussed in Chap. 4.

Finally, it was announced in [16] that the zeta functions of algebraic groups of types B_l , C_l and D_l all possessed a natural boundary, but this work is also yet to be made into print. In Chap. 5 we present a theory of natural

VIII Preface

boundaries of two-variable polynomials. This is followed by Chap. 6 where the aforementioned result on the zeta functions of classical groups is proved, and Chap. 7, where we consider the natural boundaries of the zeta functions attached to nilpotent groups listed in Chap. 2.

The first author thanks Zeev Rudnick who first informed him of Conjecture 1.11, Roger Heath-Brown who started the ball rolling and Fritz Grunewald for discussions which helped bring the ball to a stop. The first author also thanks the Max-Planck Institute in Bonn for hospitality during the preparation of this work and the Royal Society for support in the form of a University Research Fellowship. The second author thanks the EPSRC for a Research Studentship and a Postdoctoral Research Fellowship, and the first author for supervision during his doctoral studies.

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Contents

1	Introduction	1
1.1	A Brief History of Zeta Functions	1
1.1.1	Euler, Riemann	1
1.1.2	Dirichlet	3
1.1.3	Dedekind	4
1.1.4	Artin, Weil	5
1.1.5	Birch, Swinnerton-Dyer	6
1.2	Zeta Functions of Groups	6
1.2.1	Zeta Functions of Algebraic Groups	7
1.2.2	Zeta Functions of Rings	9
1.2.3	Local Functional Equations	10
1.2.4	Uniformity	11
1.2.5	Analytic Properties	12
1.3	p -Adic Integrals	14
1.4	Natural Boundaries of Euler Products	16
2	Nilpotent Groups: Explicit Examples	21
2.1	Calculating Zeta Functions of Groups	21
2.2	Calculating Zeta Functions of Lie Rings	23
2.2.1	Constructing the Cone Integral	23
2.2.2	Resolution	25
2.2.3	Evaluating Monomial Integrals	31
2.2.4	Summing the Rational Functions	32
2.3	Explicit Examples	32
2.4	Free Abelian Lie Rings	33
2.5	Heisenberg Lie Ring and Variants	34
2.6	Grenham's Lie Rings	38
2.7	Free Class-2 Nilpotent Lie Rings	40
2.7.1	Three Generators	40
2.7.2	n Generators	41
2.8	The 'Elliptic Curve Example'	42

2.9	Other Class Two Examples	43
2.10	The Maximal Class Lie Ring M_3 and Variants	45
2.11	Lie Rings with Large Abelian Ideals	48
2.12	$F_{3,2}$	51
2.13	The Maximal Class Lie Rings M_4 and Fil_4	52
2.14	Nilpotent Lie Algebras of Dimension ≤ 6	55
2.15	Nilpotent Lie Algebras of Dimension 7	62
3	Soluble Lie Rings	69
3.1	Introduction	69
3.2	Proof of Theorem 3.1	71
3.2.1	Choosing a Basis for $\mathfrak{t}_n(\mathbb{Z})$	71
3.2.2	Determining the Conditions	72
3.2.3	Constructing the Zeta Function	74
3.2.4	Transforming the Conditions	74
3.2.5	Deducing the Functional Equation	75
3.3	Explicit Examples	77
3.4	Variations	78
3.4.1	Quotients of $\mathfrak{t}_n(\mathbb{Z})$	78
3.4.2	Counting All Subrings	82
4	Local Functional Equations	83
4.1	Introduction	83
4.2	Algebraic Groups	83
4.3	Nilpotent Groups and Lie Rings	83
4.4	The Conjecture	84
4.5	Special Cases Known to Hold	86
4.6	A Special Case of the Conjecture	87
4.6.1	Projectivisation	88
4.6.2	Resolution	89
4.6.3	Manipulating the Cone Sums	91
4.6.4	Cones and Schemes	93
4.6.5	Quasi-Good Sets	95
4.6.6	Quasi-Good Sets: The Monomial Case	97
4.7	Applications of Conjecture 4.5	98
4.8	Counting Subrings and p -Subrings	102
4.9	Counting Ideals and p -Ideals	103
4.9.1	Heights, Cocentral Bases and the π -Map	104
4.9.2	Property (\dagger)	107
4.9.3	Lie Rings Without (\dagger)	119

5	Natural Boundaries I: Theory	121
5.1	A Natural Boundary for $\zeta_{\mathrm{GSp}_6}(s)$	121
5.2	Natural Boundaries for Euler Products	123
5.2.1	Practicalities	134
5.2.2	Distinguishing Types I, II and III	136
5.3	Avoiding the Riemann Hypothesis	139
5.4	All Local Zeros on or to the Left of $\Re(s) = \beta$	142
5.4.1	Using Riemann Zeros	143
5.4.2	Avoiding Rational Independence of Riemann Zeros	145
5.4.3	Continuation with Finitely Many Riemann Zeta Functions	149
5.4.4	Infinite Products of Riemann Zeta Functions	150
6	Natural Boundaries II: Algebraic Groups	155
6.1	Introduction	155
6.2	$G = \mathrm{GO}_{2l+1}$ of Type B_l	159
6.3	$G = \mathrm{GSp}_{2l}$ of Type C_l or $G = \mathrm{GO}_{2l}^+$ of Type D_l	161
6.3.1	$G = \mathrm{GSp}_{2l}$ of Type C_l	162
6.3.2	$G = \mathrm{GO}_{2l}^+$ of Type D_l	165
7	Natural Boundaries III: Nilpotent Groups	169
7.1	Introduction	169
7.2	Zeta Functions with Meromorphic Continuation	169
7.3	Zeta Functions with Natural Boundaries	170
7.3.1	Type I	171
7.3.2	Type II	171
7.3.3	Type III	173
7.4	Other Types	177
7.4.1	Types IIIa and IIIb	177
7.4.2	Types IV, V and VI	177
A	Large Polynomials	179
A.1	\mathcal{H}^4 , Counting Ideals	179
A.2	$\mathfrak{g}_{6,4}$, Counting All Subrings	180
A.3	T_4 , Counting All Subrings	180
A.4	$L_{(3,2,2)}$, Counting Ideals	181
A.5	$\mathcal{G}_3 \times \mathfrak{g}_{5,3}$, Counting Ideals	182
A.6	$\mathfrak{g}_{6,12}$, Counting All Subrings	183
A.7	\mathfrak{g}_{1357G} , Counting Ideals	184
A.8	\mathfrak{g}_{1457A} , Counting Ideals	186
A.9	\mathfrak{g}_{1457B} , Counting Ideals	187
A.10	$\mathrm{tt}_6(\mathbb{Z})$, Counting Ideals	188
A.11	$\mathrm{tt}_7(\mathbb{Z})$, Counting Ideals	188

XII Contents

B Factorisation of Polynomials Associated to Classical Groups	191
References	201
Index	205
Index of Notation	207